Modular knots from simply decorated uniform tessellations

David A. Reimann
Department of Mathematics and Computer Science
Albion College
Albion, Michigan, 49224, USA
E-mail: dreimann@albion.edu

Abstract

Knots and links are a common theme in two-dimensional artworks that span cultures and time periods. Knots are found as Greek, Roman, and Coptic decorations, reaching a high point with the Celts in the 7th century AD. A method of creating complex knots and links is presented which uses a modular approach that begins with a k-uniform tessellation. Each regular polygon is decorated with a simple motif that that has arcs connecting uniformly spaced points on the sides of the polygons. A variety of complex knots and links can be created using this procedure. Examples of visually interesting knots and links created using this procedure are presented.

1 Introduction

Knots have also been used for nearly two thousand years as decorative elements and are common in many cultures [1]. Spiral, key, and interlace patterns have been used for thousands of years as decorative elements in mosaics and other artworks. Celtic artists were masters at using plaits to decorate otherwise simple objects. While the process the Celts used for generating knots has been lost, many believe knots were created using an underlying square lattice decorated with lines connecting diagonally opposite corners.

Uniform tessellations, where the number and order of polygons meeting at a vertex remains constant throughout the tessellation, are a common decorative element for planar surfaces. The simplest uniform tessellations are the tessellations by squares, regular hexagons, and regular triangles. There are eight other uniform (more precisely 1-uniform) tessellations that form the eleven Archimedean tessellations.

The author has previously described a technique for creating interlace patterns by decorating the polygons in such tessellation using a simple motif comprised of Bézier curves [2]. Each n-gon is decorated by subdividing its sides and placing d uniformly spaced endpoints along each side, resulting in nd endpoints. Each endpoint can be assigned a unique integer $0, 1, \dots nd-1$ starting with the first side clockwise from a vertex. A arc pattern in a single polygon can be described using the following notation:

$$(\alpha, \beta | \gamma, \delta | \varepsilon, \zeta | \cdots)$$

where $\alpha, \beta, \dots$ represent the endpoint numbers of arcs in a the polygon. A total of $nd/2$ arcs made from simple cubic Bézier curves connect pairs of endpoints such that the tangent of each at the endpoints is

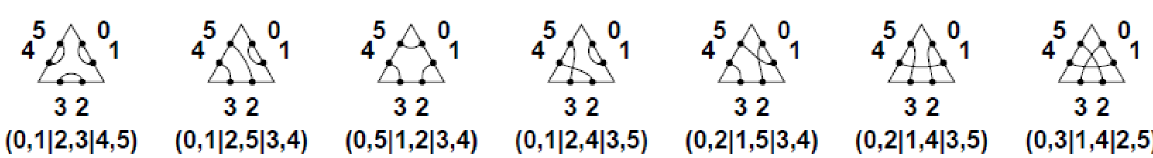


Figure 1: *Example tiles decorated with Bézier arcs. The seven geometrically unique crossing pattern motifs for decorating triangles using three Bézier arcs. Each triangle side contains two arc endpoints. Notation for each pattern is shown below the corresponding triangle. Such crossing pattern motifs can be constructed as long as the number of arc endpoints is even.*

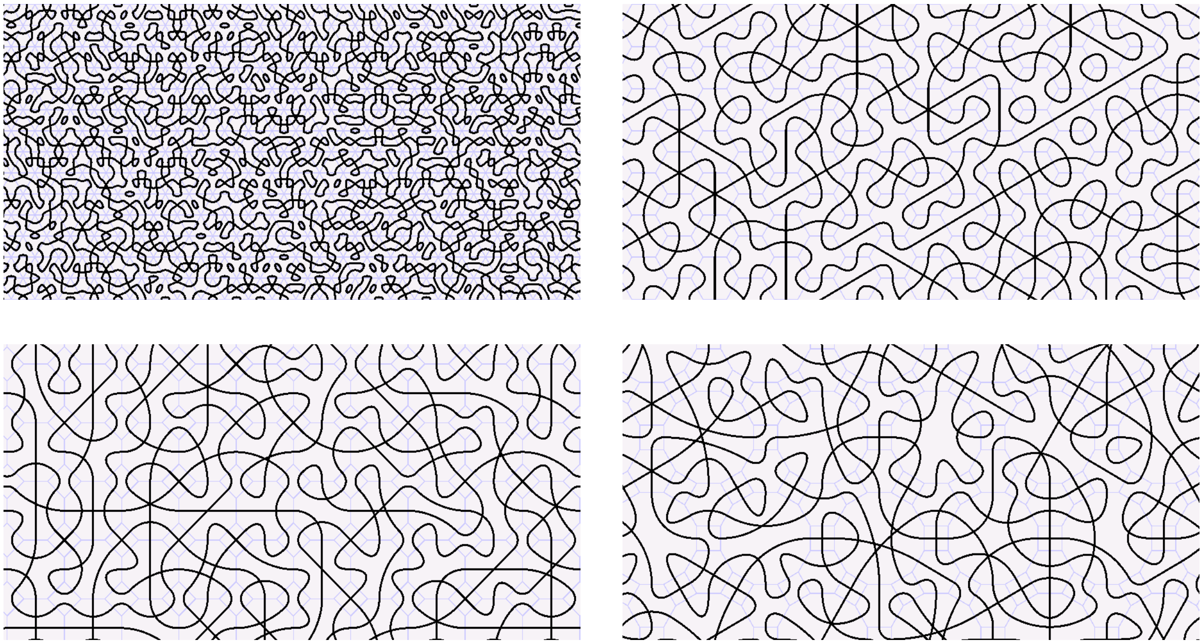


Figure 2 : *Examples of Bézier curve motifs decorating polygons comprising uniform tessellations. The upper-left figure is a tessellation by equilateral triangles (3,3,3,3,3,3) using two arc endpoints per side. The upper-right figure is a tessellation by regular hexagons (6,6,6) using one arc end point per side. The lower-left figure is a tessellation by regular octagons and squares (8,8,4) using one arc end point per side. The lower-right figure is a tessellation by regular dodecagons, hexagons, and squares (12,6,4) using one arc end point per side. The underlying tessellation is shown faintly in the background of each figure.*

perpendicular to the polygon edge. Examples of polygons decorated in this manner are shown in Figure 1. The number of unique motifs grows exponentially with the number of endpoints per side and the number of polygon sides [3]. In this manner, the curvature of the arcs at the boundary is continuous, giving a visually pleasing meandering pattern. An example using randomly placed motifs, shown in Figure 2, is suggestive of threads that wander through space in a semi-regular manner.

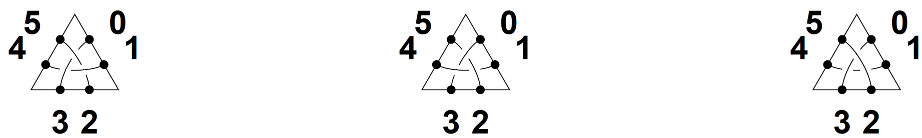
2 Methods

Knots can be generated in a modular fashion by treating the interlace patterns obtained from decorating polygons in a uniform tessellation with Bézier curves as threads that cross one another rather than simply intersecting. To indicate a crossing, the lower thread is broken into two segments near the crossing point, producing pattern commonly used in presenting knots using a knot diagram.

Finding the location where two arcs intersect requires finding the intersection point of two Bézier curves. A cubic Bézier curve is a parametric curve formed by the weighted combination of four control points, P_0, P_1, P_2, and P_3:

$$B(t) = P_0 t^3 + P_1 t^2 (1-t) + P_1 t (1-t)^2 + P_3 (1-t)^3,$$

where $0 \leq t \leq 1$. In this work, P_0 and P_3 are endpoints that lie on the edge of a polygon. The control points P_1 and P_2 are selected such that the curve is perpendicular to the edge of the polygon at points P_0 and P_3. The control points P_1 and P_2 can be selected such that the maximal curvature is also minimized along the curve. The intersection of two Bézier curves is nontrivial because in the general case, the curves may have up to six intersection points. However, since the arcs used are known to have a simple rounded shape, at most a single



$(0,3|1,4|2,5)[0/1,1/2,2/0]$ $(0,3|1,4|2,5)[0/2,1/0,2/1]$ $(0,3|1,4|2,5)[0/1,2/0,2/1]$

Figure 3: *Examples of knot sub-patterns. These over/under knot sub-patterns are based on the* $(0,3|1,4|2,5)$ *arc pattern shown in Figure 1. The arcs are labeled 0, 1, and, 2 based on the position in the parenthesized list; here arc 0 connects endpoints 0 and 3, arc 1 connects endpoints 1 and 4, and arc 2 connects endpoints 2 and 5. Note an intersection can result in either over/under crossing type. Thus a motif with k crossings can have up to* 2^k *crossing combinations (five others for this crossing pattern are not shown).*

point of intersection occurs between any given arcs. This intersection point is determined using a numerical algorithm.

The notation for an arc pattern given in the previous section can be augmented to describe a knot sub-pattern on a polygon as follows:

$$(\alpha,\beta|\gamma,\delta|\varepsilon,\zeta|\cdots)[A/B,C/D\cdots]$$

where $\alpha,\beta,\ldots$ represent endpoint numbers and A/B indicates arc A (from endpoints α to β) overcrosses arc B (from endpoints γ to δ) . Examples of such knot sub-patterns is shown in Figure 3. Note an intersection can result in either over/under crossing type, so that a motif with k crossings can have up to 2^k crossing combinations.

The number of crossing combinations results in nontrivial drawing of knot sub-patterns. This can be seen in the arcs shown in Figure 3. In some cases, an arc can be drawn as a continuous segment between endpoints, such as arc 2 (between endpoints 2 and 5) in the pattern $(0,3|1,4|2,5)[0/1,2/0,2/1]$. However in this example, arc 1 (between endpoints 1 and 4) must be drawn as three separate sub-arcs.

Bain described a process for creating Celtic knots by connecting diagonally opposite corners in square lattice decorated [4]. The dual of a lattice of squares results in new lattice of squares rotated 45° from the original, so that a diagonal line passing through vertices in the original lattice will pass through the midpoints of edges in its dual as seen in Figure 4. Thus, the technique described here can be though of as a generalization of such Celtic knots.

3 Results

Examples using this procedure were constructed on several tessellations. Figure 5 shows an example containing several knot patterns based on the tessellation by triangles (3,3,3,3,3,3) where each triangle is decorated with three arcs. Figure 6 shows an example containing a repeating knot pattern based on a tessellation by squares (4,4,4,4) where each square is decorated with four arcs. Figures 7 and 8 show examples containing repeating knot patterns based on tessellations by hexagons (6,6,6) where each hexagon is decorated with three arcs. Figure 9 shows an example containing a several knot pattern based on a tessellation by octagons and squares (8,8,4) where each polygon is decorated with arcs connecting edge midpoints. Figure 10 shows an example containing a several knot pattern based on a 2-uniform tessellation by dodecagons, squares, and triangles, (12,12,3;12,3,4,3) where each polygon is decorated with arcs connecting edge tri-points.

4 Discussion

Even with this very simple design element, a wealth of interesting knots patterns is possible. One reason for the visual appeal of the patterns is the arcs of adjacent tiles are not only continuous, but also have a continuous first derivative resulting in a visually smooth transition regardless of tile orientation. Minimizing the maximal curvature of the arcs within a triangle results in curves with graceful sweeps.

Future work involves improving the three-dimensional display of these knots. Another goal is to identify and characterize the knots and links present in a given pattern. Given a tessellation and motif family, it is unknown how many knot diagrams for a given knot are constructable for a given knot, this is of particular interest for knots that can be presented in a symmetric manner. It is also of interest to analyze knots by calculating invariants.

A method was described for constructing visually interesting knots in a modular manner based on k-uniform tessellations of the plane. These patterns can be useful in very large field architectural tilings.

5 Acknowledgments

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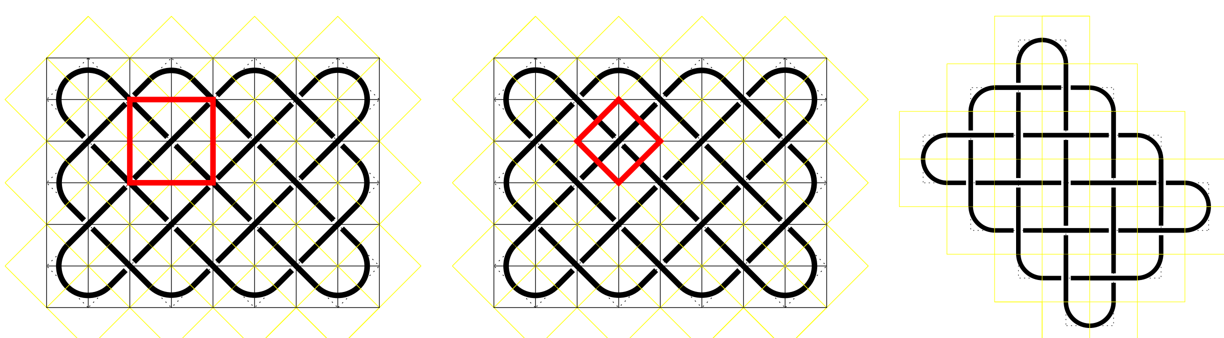

Figure 4 : *An example Celtic-style knot. Celtic knots such as seen in this figure can be considered a special case of the knot patterns described in this text. By considering the dual tessellation of the original square lattice and rotating by* 45°*, the rightmost pattern can be seen to be comprised of squares decorated with arcs connecting midpoints of edges.*

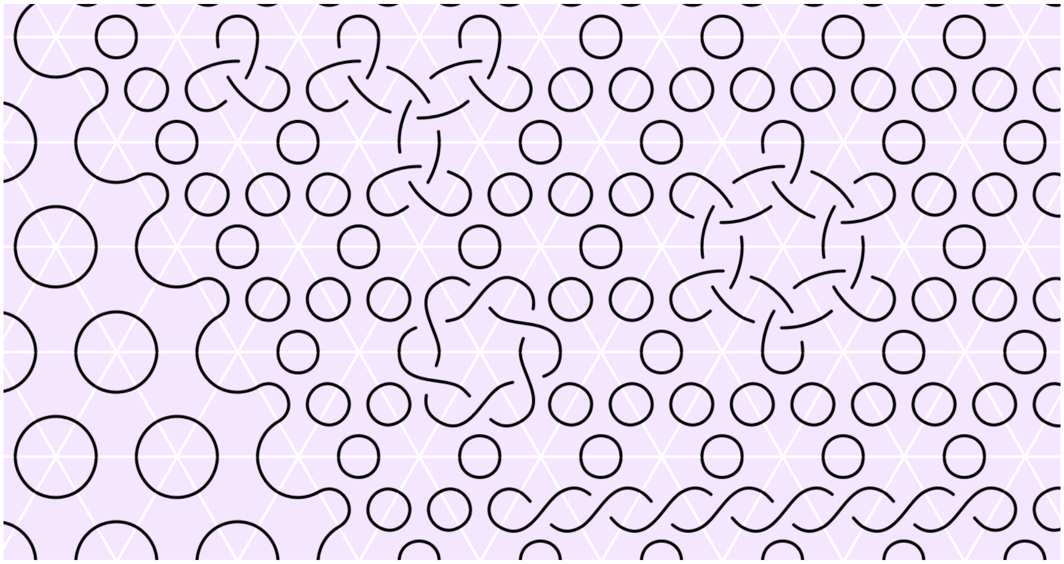

Figure 5 : *An example based on the tessellation by triangles. This figure shows an example containing several knot patterns based on a triangular tessellation (3,3,3,3,3,3) where each triangle is decorated with three arcs. Note that trefoil knots can be constructed in this manner.*

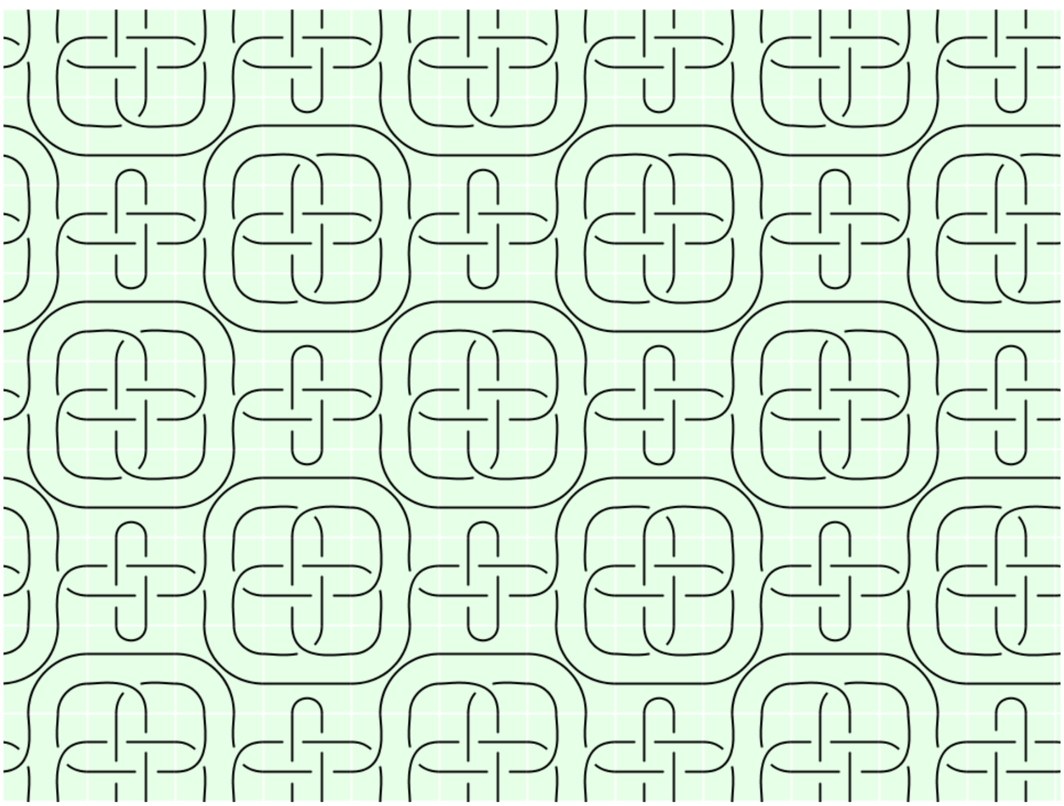

Figure 6 : *An example based on the tessellation by squares. This figure shows an example containing a repeating knot pattern based on a tessellation by squares (4,4,4,4) where each square is decorated with four arcs.*

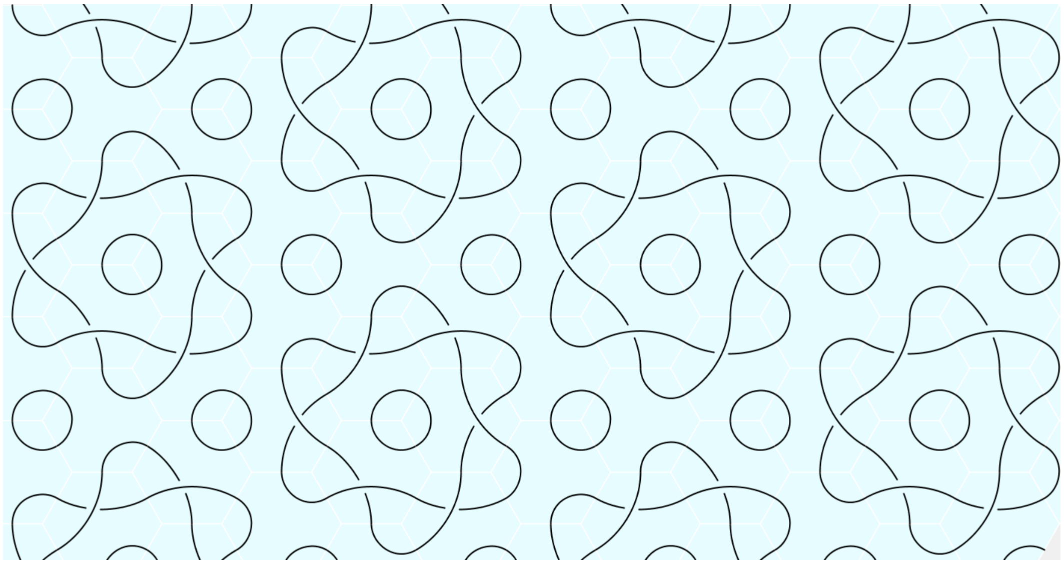

Figure 7 : *An example based on the tessellation by hexagons. This figure shows an example containing a repeating knot pattern based on a tessellation by hexagons (6,6,6) where each hexagon is decorated with three arcs.*

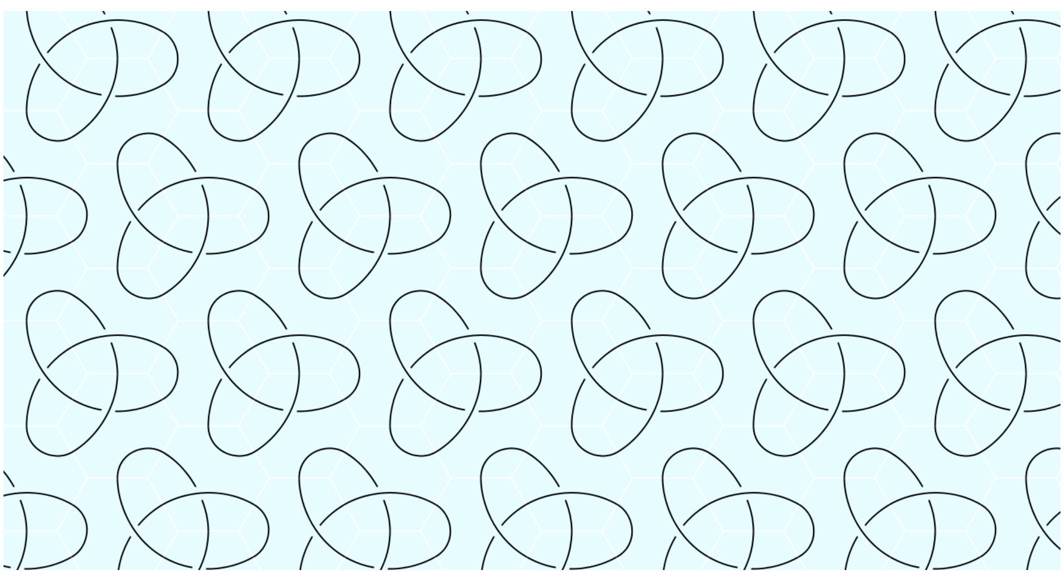

Figure 8 : *An example based on the tessellation by hexagons. This figure shows an example containing a repeating pattern of trefoil knots based on a tessellation by hexagons (6,6,6) where each hexagon is decorated with three arcs.*

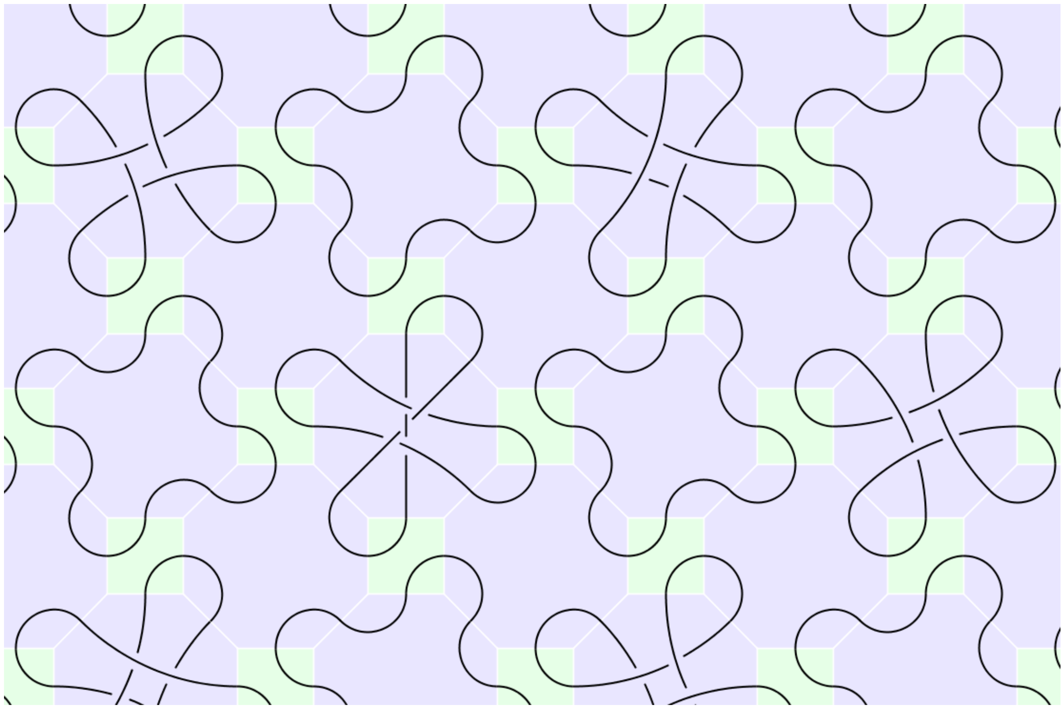

Figure 9 : *An example based on the tessellation by octagons and squares. This figure shows an example containing a several knot pattern based on a tessellation by octagons and squares (8,8,4) where each polygon is decorated with arcs connecting edge midpoints.*

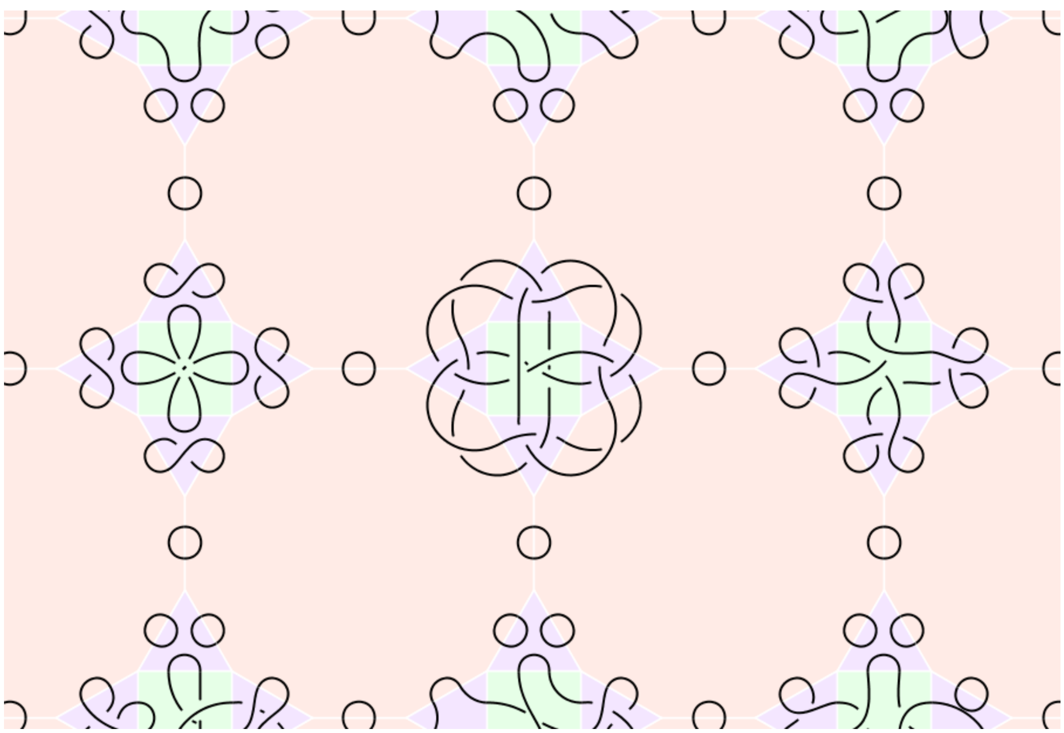

Figure 10 : *An example based on a 2-uniform tessellation. This figure shows an example containing a several knot pattern based on a 2-uniform tessellation by dodecagons, squares, and triangles, (12,12,3;12,3,4,3) where each polygon is decorated with arcs connecting edge tri-points.*

