An investigation of point symmetry patterns on the regular hexagonal tessellation is presented. This tessellation has three point symmetry groups. However, the restriction to the hexagonal tessellation causes some symmetry subgroups to be repeated in ways that are geometrically unique and others that are geometrically equivalent, resulting in a total of 14 geometrically distinct symmetry groups. Each symmetry group requires a particular set of motif symmetries to allow its construction. Examples of symmetric patterns are shown for several simple motif families.

Introduction

Throughout history, the symmetry found in regular polygons has been used to created interesting patterns. These symmetries have been mathematically analyzed and give rise to an important class of groups, namely the dihedral groups. The symmetry group of a regular polygon with \( n \) sides is given by the dihedral group of order \( 2n \), denoted \( D_n \) [2] and \( *n \) using orbifold notation [1]. This group contains elements that correspond to rotations of the polygon and rotations following a reflection about a line through the center and one of the vertices. The cyclic groups, \( C_n \), which contain only rotations, are important subgroups of dihedral groups.

Uniform tessellations, where the number and order of regular polygons meeting at a vertex remains constant throughout the tessellation, are a common decorative element for planar surfaces. The simplest uniform tessellations are the tessellations by squares, regular hexagons, and regular triangles. Each uniform tessellation of the plane by undecorated tiles has an underlying wallpaper or planar symmetry group. The tessellation of the plane using regular hexagons has the \( *632 \) wallpaper symmetry group using orbifold notation [1]. In this notation, the \( * \) indicates the symmetry has points with dihedral symmetry. The numbers following \( * \) indicate the order of the symmetry, with 6 indicating a dihedral group of order twelve (\( D_6 \)), 3 indicating a dihedral group of order six (\( D_3 \)), and 2 indicating a dihedral group of order four (\( D_2 \)). The centers of each of these point symmetry groups are located at unique locations with respect to the hexagons as shown in Figure 2. The \( D_6 \) group fixes a point at the center of the hexagon, the \( D_3 \) group fixes a point at a vertex, and the \( D_2 \) group fixes a point located at the midpoint of the edge of a hexagon.

By decorating the tiles in a tessellation with simple motifs, one can create elaborate patterns in a modular manner. For example, the author has previously described a technique for creating interlace patterns by decorating the polygons in a regular tessellation using a simple motif using Bézier curves [3]. Each \( n \)-gon is decorated by simple cubic Bézier curve connecting pairs of edge midpoints. The set of the possible geometrically unique motifs (unique up to rotation) for decorating a hexagon with three arcs is shown in Figure 1. Decorating tiles with such motifs can also alter the symmetry group of a tessellation. This work presents the possible point symmetry groups possible for the regular hexagonal tessellation.

Methods and Results

Creating a point symmetric pattern from decorated regular hexagonal tiles depends on the symmetry groups of the hexagonal tessellation and the available tile decorations. The subgroup structure of the three dihedral
groups found in the hexagonal tessellation \((D_6, D_3, \text{and} D_2)\) give all possible symmetry groups. While multiple subgroups of the same type are present, this does not result in a repeated symmetry pattern in all cases. For example, the \(C_3\) subgroup of \(D_6\), denoted here as \(D_6 : C_3\), fixes a point at the center of the hexagon while the \(C_3\) subgroup of \(D_3\), denoted \(D_3 : C_3\), fixes a point at a vertex, resulting in two distinct symmetry patterns on the hexagonal tessellation. Likewise, the six \(D_1\) subgroups of \(D_6\) having bilateral symmetry fall into two geometric classes: the groups that reflect about a line through the midpoints of edges \((D_1E)\) and the groups that reflect about a line through opposite vertices \((D_1V)\). Of the 27 total subgroups (16 for \(D_6\), 6 for \(D_3\), and 5 for \(D_2\)), there are only 14 geometrically unique symmetry patterns for the hexagonal tessellation as shown in Figure 2.

Some point symmetry patterns require individual hexagonal tiles having specific tile patterns. For example, \(D_6 : C_3\) requires a center tile with general \(C_3\) symmetry, while \(D_3 : C_3\) has no such requirement. Similarly, \(D_6 : D_6\) requires a center tile with \(D_6\) symmetry, field tiles with both \(D_1V\) symmetry and \(D_1E\) symmetry because there are mirror lines that bisect hexagons at edge midpoints and through opposite vertices. In the subgroups of the \(D_6\) symmetry group, the motif of the center polygon must be selected to match the subgroup symmetry. In outer polygons where the center lies along a mirror line, the polygon motif must have \(D_1\) symmetry: \(D_1V\) symmetry if the mirror line also passes through a vertex and \(D_1E\) if the mirror line passes through an edge midpoint. When selecting a motif, its symmetry type must be verified to match the requirements of a particular location. Each motif must also be rotated to match the subgroup symmetry of the tile location. For example, tiles surrounding the center tile in the \(D_6 : D_6\) pattern require \(D_1E\) symmetry and must be oriented correctly with respect to the mirror lines.

Figure 3 shows two examples with \(C_6\) symmetry using related motif families. As with the previous examples, these patterns were created by drawing motif patterns on a symmetric hexagonal grid. The selected symmetry has a corresponding origin that is used to determine the set of equivalent hexagons from the standpoint of the symmetry group. The motif patterns were then selected for one tile and mapped via the symmetry operation to all other equivalent hexagons.

**Discussion**

The regular hexagonal tessellation has three parent point symmetry groups with a total of 27 subgroups. However, the restriction to the hexagonal tessellation causes some symmetry subgroups to be repeated,
Figure 2: Example patterns using the geometrically unique symmetry groups on the tessellation by regular hexagons. In the *632 symmetry group, the center of each hexagon is the center point of the dihedral group $D_6$, each vertex is the center point of the dihedral group $D_3$, and each edge midpoint is the center point of the dihedral group $D_2$. Mirror lines are shown along with the orbit of an example point in the given symmetry group $G:H$, where $G$ is the parent group ($D_6, D_3, D_2$) and $H$ is the subgroup. The number of equivalent symmetries is shown in parentheses for each point symmetry group. The unique tile motifs (see Figure 1) required for each symmetry group are shown following the semicolon in each pattern name.
resulting in only 14 geometrically unique point symmetric groups. Example point symmetric patterns for each of these 14 point symmetry groups were given. Note that there are two geometrically unique forms of $D_1$ symmetry. While these patterns look similar, $D_6 : D_{1a}$ requires tiles that have $D_{1v}$ symmetry and $D_6 : D_{1b}$ requires tiles that have $D_{1E}$ symmetry. Likewise, there are three geometrically unique forms of $D_3$ symmetry and two geometrically unique forms of $D_2$ symmetry. Each of the point symmetry groups requires a particular set of motif symmetries to allow its construction. Any motif pattern family having tiles with appropriate symmetries can be used to create symmetry patterns. The bounded nature of symmetric patterns created in this manner can have more visual appeal than conventional planar tessellations.

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References