A review of Gauss's 3/23/1835 talk on quadratic functions

A. Student

January 20, 2021

Overview

On March 23, 1835 the eminent mathematician Carl Friedrich Gauss came to Albion College and gave a colloquium talk entitled "Quadratic functions: not just for squares" [1]. In his talk he gave a detailed overview of quadratic functions and explained some related mathematical concepts. He then presented several interesting applications. Gauss is pictured in Figure 1.

Figure 1: Johann Carl Friedrich Gauss
http://www-history.mcs.st-and.ac.uk/PictDisplay/Gauss.html

Summary

A quadratic function f is a polynomial function of the form

$$f(x) = ax^2 + bx + c, \tag{1}$$

where $x, a, b,$ and c are traditionally real numbers and $a \neq 0$. Note that f is simply a polynomial of degree 2. The name quadratic has been used since at least 1647 and comes from the Latin word *quadraticus*, meaning square. When $a = 0$ and $b \neq 0$, then the corresponding function is linear rather than quadratic. The form given in equation 1 is called standard form. Other common representation forms include vertex form,

$$f(x) = a(x-h)^2 + k \tag{2}$$

and factored form,

$$f(x) = a(x-r_1)(x-r_2). \tag{3}$$

The form used often depends on a particular application and converting from one form to another is very useful. The graph of f is a parabola with the vertex at the point (h, k).

Gauss showed us that the vertex form (equation 2) can be obtained from standard form (equation 1) using the technique of "completing the square." In particular,

$$h = -\frac{b}{2a} \qquad \text{and} \qquad k = -\frac{b^2 - 4ac}{2a}.$$

To find the factored form (equation 1), one substitutes the above representations for h and k into the vertex form (equation 2), resulting in

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{2a}. \tag{4}$$

Setting equation 4 equal to zero and algebraically solving for x results in

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \tag{5}$$

The two values for x are the values r_1 and r_2.

Gauss then explained that the values r_1 and r_2 are called the roots of the polynomial. If they are both real, then they represent to points on the graph of $f(x)$ where the function crosses the x-axis. It may be possible that $r_1 = r_2$, and in that case, the graph is tangent to the x-axis. If the roots are not real, then they are complex conjugate pairs, meaning $r_1 r_2$ is real. Because f is a polynomial, the chain rule can be used to find the first and second derivatives, with $f'(x) = 2ax+b$ and $f''(x) =$

$2a$. Thus the sign of a tells us the orientation of the graph at the vertex.

Gauss illustrated these concepts with the quadratic function

$$\begin{aligned} f(x) &= x^2 - 2x - 8, \\ &= (x-1)^2 - 9, \qquad h = 1, k = -9, \\ &= (x+2)(x-4), \qquad r_1 = -2, r_2 = 4. \\ f'(x) &= 2x - 2 \quad \text{and} \\ f''(x) &= 2. \end{aligned}$$

Thus the function crosses the axis at the points -2 and 4, is concave up, and has a vertex at the point $(1, -9)$ as illustrated in Figure 2.

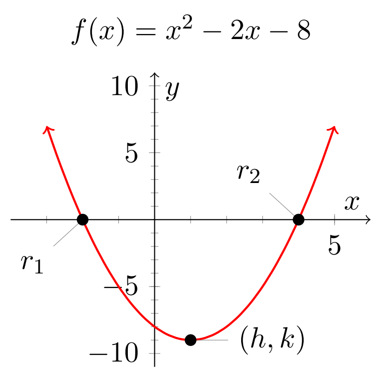

Figure 2: Example quadratic function
The function $f(x) = x^2 - 4x - 8$ has roots $r_1 = -2$, $r_2 = 4$, and vertex $(1, -9)$.

Question and answer

After the talk, I asked Gauss "Can a general polynomial be expressed in factored form?" He responded about his earlier work showing every polynomial of degree n can be factored into exactly n linear expressions of the form $(a_i x - b_i)$, where the a_i and b_i terms are possible complex numbers. He also mentioned while general cubic and quartic equations can be solved exactly, other mathematicians had recently shown that the general quintic and higher order polynomials have no closed form solution. Finally, he said Newton had developed an iterative method for numerically finding roots of a polynomial using its derivative. Let x_0 be an estimate of a root, then form the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Then x_n will generally approach a root, say r_1, under certain conditions, so that

$$\lim_{n\to\infty} f(x_n) = 0.$$

One can divide the original polynomial by $x - r$ using polynomial division, yielding

$$g_1(x) = \frac{f(x)}{x - r_1}.$$

In theory this process can be repeated until all n roots are determined. However, numerical errors may limit the practical application of this technique for polynomials of large degree.

Bring a friend

I brought my friend Charles F. Stockwell[1] to the talk.

Personal response

I really enjoyed the talk by Gauss. First, it was exciting to have such an eminent mathematician visit Albion and have the opportunity to meet him. I was surprised that he was here all the way from his home in Germany. While I had seen some of the material presented, much of it was new. I am very interested in taking *Abstract Algebra* and learning more about group theory and its relationship to finding polynomial roots.

References

[1] Johann Carl Friedrich Gauss. Quadratic functions: not just for squares. Albion College Mathematics and Computer Science Colloquium. 23 March 1835.

[1] Albion's first Principal and the equivalent of today's president. Tragically he died and was buried at sea in 1850.

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% Note that % is a comment, the remaining text on the line is ignored.

\documentclass[11pt,letterpaper,twocolumn]{article} % LaTeX Article Template
% uses 11 point font on 8.5x11 paper
% and a two column format

\usepackage{listings}

\usepackage[T1]{fontenc}
\usepackage[english]{babel}
\usepackage{fontawesome}
\usepackage[colorlinks=true, urlcolor=blue, citecolor=black, linkcolor=black]{hyperref}


\usepackage{url}
\usepackage{graphicx}
\usepackage{amsmath}
\usepackage[top=.75in, bottom=.75in, left=.5in, right=.5in]{geometry}

\usepackage{nopageno}

\def\BibTeX{{\rm B\kern-.05em{\sc i\kern-.025em b}\kern-.08em
               T\kern-.1667em\lower.7ex\hbox{E}\kern-.125emX}}


\usepackage{tikz}
\usepackage{pgfplots}
\usetikzlibrary{decorations.text}
\usetikzlibrary{shapes}
\begin{document}                    % Set the beginning of a LaTeX document



% ================ Change information below this line ==============

% Use a title withe the following format for speaker reviews
\title{A review of Gauss's 3/23/1835 talk \\on quadratic functions}
\author{A.\ Student}
\date{\today}
\maketitle

\section*{Overview}   % REQUIRED 50-100 words
% Get this information from speaker introduction and talk abstract

On March 23, 1835 the eminent mathematician Carl Friedrich Gauss came to Albion College
and gave a colloquium talk entitled ``Quadratic functions: not just for squares'' \cite{Gauss}.
In his talk he gave a detailed overview of quadratic functions and explained some related mathematical concepts.
He then presented several interesting applications.
Gauss is pictured in Figure~\ref{fig:Gauss}.

\begin{figure}[h]
        \begin{center}
                \includegraphics[width=1.5in]{Gauss.jpg}
        \end{center}
        \caption{Johann Carl Friedrich Gauss}
        {\footnotesize \url{http://www-history.mcs.st-and.ac.uk/PictDisplay/Gauss.html}}
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        \label{fig:Gauss}
\end{figure}


\section*{Summary} % REQUIRED 100-400 words

% Important Point 1 of 3

A quadratic function $f$ is a polynomial function of the form
\begin{equation}f(x) = a x^2 + b x + c,
\label{eq:quadS}
\end{equation}
where $x, a, b, \text{and } c$ are traditionally real numbers and $a\ne0$.
Note that $f$ is simply a polynomial of degree 2.
The name quadratic has been used since at least 1647 and comes from the Latin word \emph{quadraticus}, meaning square.
When $a=0$ and $b\ne 0$, then the corresponding function is linear rather than quadratic.
The form given in equation~\ref{eq:quadS} is called standard form. Other common representation forms include
vertex form,
\begin{equation}f(x) = a (x-h)^2 + k
\label{eq:quadV}
\end{equation}
and factored form,
\begin{equation}f(x) = a(x-r_1)(x-r_2).
\label{eq:quadF}
\end{equation}
The form used often depends on a particular application and converting from one form to another is very useful.
The graph of $f$ is a parabola with the vertex at the point $(h,k)$.

% Important Point 2 of 3

Gauss showed us that the vertex form (equation~\ref{eq:quadV}) can be obtained
from standard form (equation~\ref{eq:quadS})
using the technique of ``completing the square.'' In particular,
\begin{equation*}
h = -\frac{b}{2a} \qquad\text{and}\qquad k = -\frac{b^2-4ac}{2a}.
\end{equation*}
To find the factored form (equation~\ref{eq:quadS}), one
substitutes the above representations for $h$ and $k$ into the vertex form (equation~\ref{eq:quadV}), resulting in
\begin{equation}
f(x) = a \left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{2a}.
\label{eq:quadV2}
\end{equation}
Setting equation~\ref{eq:quadV2} equal to zero and algebraically solving for $x$
results in
\begin{equation}
x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.
\label{eq:quadRoots}
\end{equation}
The two values for $x$ are the values $r_1$ and $r_2$.

% Important Point 3 of 3

Gauss then explained that the values $r_1$ and $r_2$ are called the roots of the polynomial.
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If they are both real, then they represent to points on the graph of $f(x)$ where the function
crosses the $x$-axis. It may be possible that $r_1=r_2$, and in that case, the graph is tangent
to the $x$-axis.
If the roots are not real, then they are complex conjugate pairs, meaning $r_1r_2$ is real.
Because $f$ is a polynomial, the chain rule can be used to find the first and second derivatives, with $f'(x) = 2ax + b$ and
$f''(x) = 2a$. Thus the sign of $a$ tells us the orientation of the graph at the vertex.

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\vspace*{-2\baselineskip}
\begin{align*}
f(x) &= x^2 - 2x - 8, \\
&= (x-1)^2 - 9,\qquad h=1, k=-9,\\
&= (x+2)(x-4),\qquad r_1=-2, r_2=4.\\
f'(x) &= 2x - 2\quad\text{and}\\
f''(x) &= 2.\\
\end{align*}

\vspace*{-2\baselineskip}
Thus the function crosses the axis at the points -2 and 4, is concave up, and has a vertex at the point $(1,-9)$ as illustrated in Figure~\ref{fig:quadratic}.

\begin{figure}[h]
	\begin{center}
		\pgfplotsset{width=2.5in}
		\begin{tikzpicture}[smooth]
		\begin{axis}[xmin=-4,xmax=6,ymin=-11,ymax=11,
		title={$f(x) = x^2 - 2x - 8$},
		axis lines=middle,
		axis line style={->},
		xtick={-5,0,5},
		minor xtick={-3,...,5},
		minor ytick={-9,...,9},
		xlabel=$x$,
		ylabel={$y$}
		]
		\addplot[red,thick,<->,domain=-3:5] {x^2 - 2*x - 8};
		\addplot[only marks,mark=*] coordinates {(-2,0) (4,0) (1,-9)};
		\node[coordinate,pin=below left:{$r_1$}] at (axis cs:-2,0) {};
		\node[coordinate,pin=above left:{$r_2$}] at (axis cs:4,0) {};
		\node[coordinate,pin=right:{$(h,k)$}] at (axis cs:1,-9) {};
		\end{axis}
		\end{tikzpicture}
	\end{center}


	\caption{Example quadratic function}
	The function $f(x) = x^2 - 4x - 8$ has roots $r_1=-2$, $r_2 = 4$,
	and vertex $(1,-9)$.
	\label{fig:quadratic}
\end{figure}


\section*{Question and answer} % Optional 50-100 words

After the talk, I asked Gauss ``Can a general polynomial be expressed in factored form?''
```

```latex
He responded about his earlier work
showing every polynomial of degree $n$ can be factored into exactly $n$ linear expressions of the form $(a_i x - b_i)$, where
the $a_i$ and $b_i$ terms are possible complex numbers.
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other mathematicians had recently shown that the general quintic and higher order polynomials have no closed form solution.
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Then $x_{n}$ will generally approach a root, say $r_1$, under certain conditions, so that
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However, numerical errors may limit the practical application of this technique
for polynomials of large degree.


\section*{Bring a friend} % Optional 50-100 words

I brought my friend Charles F. Stockwell\footnote{Albion's first Principal and the equivalent of today's president.
       Tragically he died and was buried at sea in 1850.} to the talk.


%\section*{Solutions to posed problems} % Optional


\section*{Personal response} % REQUIRED 50-100 words

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and have the opportunity to meet him. I was surprised that he was here all the way from his home in Germany.
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and learning more about group theory and its relationship to finding polynomial roots.




%\raggedright - consider using if URL Spacing looks ugly
% Bibliography - - - REQUIRED
\bibliographystyle{albion} % plain citing, requires the file albion.bst
\bibliography{colloquium} % uses colloquium.bib



\onecolumn

% =============== LaTeX Source - Always include! =============
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        columns=flexible,
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% ================ Grading Rubric - Always include! ==============
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\noindent
\begin{center}
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        \scriptsize\sffamily
        \begin{tabular}{|c||c|c|c|c|c|c|c||c|c||c|}
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                \textbf{Grading}&\multicolumn{7}{|c||}{\textbf{Required}}
                & \multicolumn{2}{c||}{\textbf{Optional}} & \textbf{Total} \\\cline{2-11}%
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                \textbf{Rubric}&Title &Intro & Summary & Reflection & Writing & \LaTeX\ \href{http://zeta.albion.edu/~dreimann/Spring2021/courses/math-cs-299-399/TeX-Os.pdf}{\scriptsize\faExternalLink} & \BibTeX & Q\&A & Solutions & 0 1 2 3 4 5 6 7 9 10 11 12 \\
                \href{http://zeta.albion.edu/~dreimann/Spring2021/courses/math-cs-299-399/reviews.php}{\scriptsize\faExternalLink}&
                0 1  & 0 1  & 0 1 2 3 & 0 1  & 0 1 2 & 0 1 2 & 0 1 2  & 0 1 2  & 0 1 2 3 4      & 13 14 15 16 17 18 \\ \hline
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Grading Rubric	**Required**							**Optional**		**Total**
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